An Explicit Rate-Optimal Streaming Code for Channels with Burst and Arbitrary Erasures

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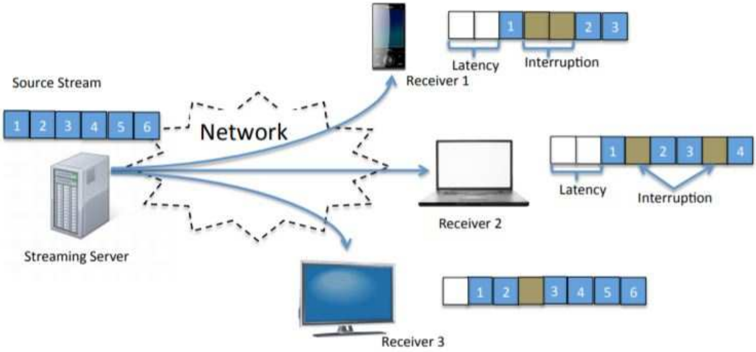
Multimedia Streaming



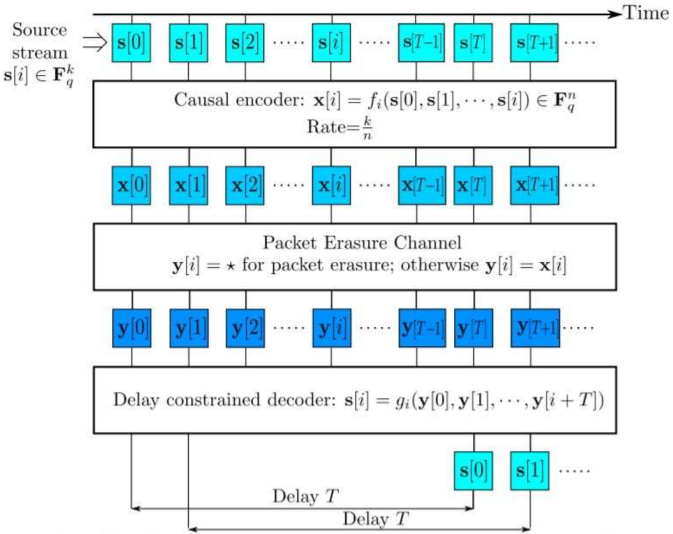
Application	Delay
Voice.	150 ms
Vidoe Conf.	100 ms
Gaming	50 ms

In most cases retransmission does not meet the delay constraint

Problem Setup

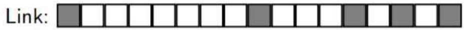
- **Source Model** : i.i.d. sequence $s[t] \sim p_s(\cdot) = Unif\{(\mathbb{F}_q^k)\}$
- **Streaming Encoder**: $x[t] = f_t\,(s[0], \ldots, s[t])$, $x[t] \in (\mathbb{F}_q^n)$
- Erasure Channel (To be specified)
- **Delay-Constrained Decoder**: $\hat{s}[t] = g_t(y[0], \ldots, y[t+T])$
- Rate $R = \frac{k}{n}$

Real-Time Communication System

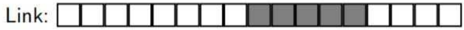


Streaming Code: Causal Encoder + Delay Constrained Decoder

Channel Model

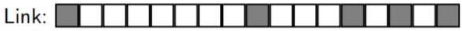


(a): I.I.D. Erasure Sequence

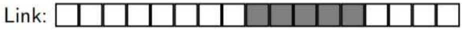


(b): Burst Erasure Sequence

Channel Model



(a): I.I.D. Erasure Sequence



(b): Burst Erasure Sequence

Capacity (T=delay, N=Arbitrary erasures, B=Burst erasures)

- For any $T \geq B \geq N = 1$ (Martinian et al. (04))

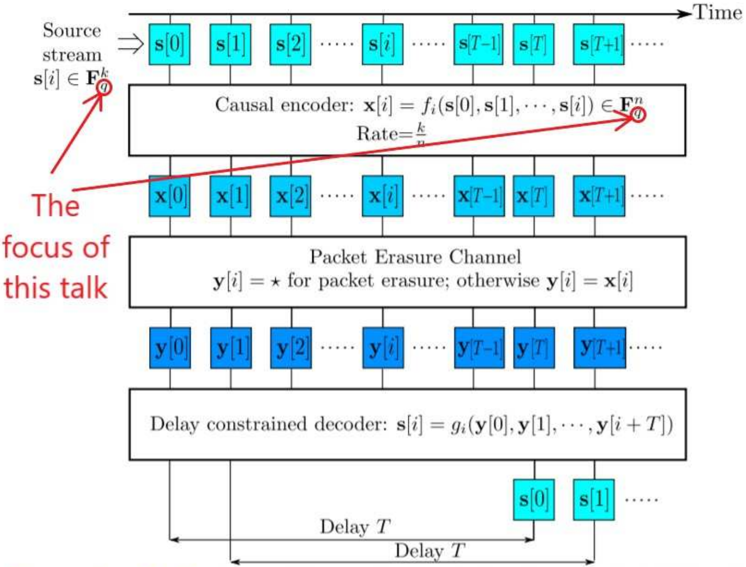
$$C(T, B, 1) = \frac{T}{T + B}$$

- For any $T \geq B \geq N$, $N > 1$ (Fong et al. (18), Krishnan et al. (18))

$$C(T, B, N) = \frac{T - N + 1}{T - N + B + 1}$$

- Can be extended to the sliding window model (Badr et al. (13))

Real-Time Communication System



Streaming Code: Causal Encoder + Delay Constrained Decoder

Achievable Schemes

For any $T \geq B \geq N = 1$

- Explicit ✓
- Field size: scales linearly with the delay ($O(T)$) ✓

For any $T \geq B \geq N$, $N > 1$

- Explicit ?
- Field size ?

Achievable Schemes

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For any $T \geq B \geq N$, $N > 1$

- Explicit ?
- Field size ?

General construction

- Block code at rate $C(T, B, N)$ with a delay-constraint T (symbol-level)
- Diagonal interleaving
  - Originally suggested by Martinian et al. (04) for burst only channels
  - Recently extended to general channels by Krishnan et al. (19)

Example: $T = 3$, $B = 2$, $N = 1$ (Martinian and Sundberg)

- Step 1: Rate 3/5 block code

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$[a, b, c] \longrightarrow [a, b, c, a + c, b + c]$$

- Step 2: Diagonal interleaving

$x[i-1]$	$x[i]$	$x[i+1]$	$x[i+2]$	$x[i+3]$	$x[i+4]$
a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}
b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}
c_{i-1}	c_i	c_{i+1}	c_{i+2}	c_{i+3}	c_{i+4}
$a_{i-4} + c_{i-2}$	$a_{i-3} + c_{i-1}$	$a_{i-2} + c_i$	$a_{i-1} + c_{i+1}$	$a_i + c_{i+2}$	$a_{i+1} + c_{i+3}$
$b_{i-4} + c_{i-3}$	$b_{i-3} + c_{i-2}$	$b_{i-2} + c_{i-1}$	$b_{i-1} + c_i$	$b_i + c_{i+1}$	$b_{i+1} + c_{i+2}$

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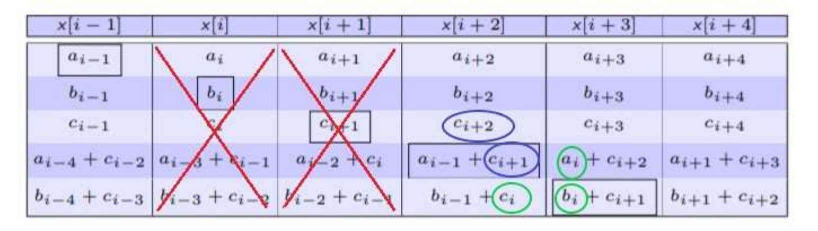

$x[i-1]$	$x[i]$	$x[i+1]$	$x[i+2]$	$x[i+3]$	$x[i+4]$
a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}
b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}
c_{i-1}	c_i	c_{i+1}	c_{i+2}	c_{i+3}	c_{i+4}
$a_{i-4}+c_{i-2}$	$a_{i-3}+c_{i-1}$	$a_{i-2}+c_i$	$a_{i-1}+c_{i+1}$	a_i+c_{i+2}	$a_{i+1}+c_{i+3}$
$b_{i-4}+c_{i-3}$	$b_{i-3}+c_{i-2}$	$b_{i-2}+c_{i-1}$	$b_{i-1}+c_i$	b_i+c_{i+1}	$b_{i+1}+c_{i+2}$

Can recover from from a burst of two erasures

Example: $T = 3,\ B = 2,\ N = 1$ (Martinian and Sundberg)

- Step 1: Rate 3/5 block code

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$[a, b, c] \longrightarrow [a, b, c, a+c, b+c]$$

- Step 2: Diagonal interleaving

$x[i-1]$	$x[i]$	$x[i+1]$	$x[i+2]$	$x[i+3]$	$x[i+4]$
a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}
b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}
c_{i-1}	c_i	c_{i+1}	c_{i+2}	c_{i+3}	c_{i+4}
$a_{i-4}+c_{i-2}$	$a_{i-3}+c_{i-1}$	$a_{i-2}+c_i$	$a_{i-1}+c_{i+1}$	a_i+c_{i+2}	$a_{i+1}+c_{i+3}$
$b_{i-4}+c_{i-3}$	$b_{i-3}+c_{i-2}$	$b_{i-2}+c_{i-1}$	$b_{i-1}+c_i$	b_i+c_{i+1}	$b_{i+1}+c_{i+2}$

Can not recover from more than one sporadic erasure...

Achievable Schemes For $N > 1$: What is Known?

Construction	Field size	Explicit
Fong et al. (18)	$O\binom{T}{N}$	No
Dudzicz et al. (19)	$O(exp(T))$	Yes (for $R \geq \frac{1}{2}$)
Krishnan et al. (18)	$O(\exp(T))$	Yes
Krishnan et al. (19)	$O(T^2)$	**only for specific cases**

Can we find an **explicit** capacity achieving code with field size that **scales quadratically** with the delay constraint ($O(T^2)$)?

Suggested Construction

Step 1

- Take an (n, k) MDS code $\mathcal{C}''$ over $\mathbb{F}_q$ with the generator matrix

$$\mathbf{G}'' = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & Y & \cdots & \cdots & \cdots & \cdots & Y \\ 0 & 1 & 0 & 0 & \cdots & 0 & Y & \cdots & \cdots & \cdots & \cdots & Y \\ 0 & 0 & \ddots & 0 & \cdots & 0 & Y & \cdots & \cdots & \cdots & \cdots & Y \\ \vdots & \vdots & & 1 & & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & & & \ddots & 0 & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 & Y & \cdots & \cdots & \cdots & \cdots & Y \end{bmatrix}$$

- Y is a place-holder

Suggested Construction

Step 2

- Perform row operations to generate

$$\mathbf{G}' = \underbrace{\begin{bmatrix} 1 & X & \cdots & X & 0 & 0 & 0 & \cdots & 0 & X & \cdots & X \\ 0 & 1 & X & \cdots & X & 0 & 0 & \cdots & 0 & \vdots & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 & \cdots & 0 & X & \cdots & X \\ \vdots & \vdots & & 1 & \ddots & \ddots & \ddots & & \vdots & X & & X \\ \vdots & \vdots & & & \ddots & X & \cdots & X & 0 & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 & X & \cdots & X & X & \cdots & X \end{bmatrix}}_{k \quad\quad N-1 \quad\quad B-N+1}$$

Suggested Construction

Step 3

$$\mathbf{G} = \left[\underbrace{\begin{matrix} 1 & X & \cdots & X & 0 & 0 \\ 0 & 1 & X & \cdots & X & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & & 1 & \ddots & \ddots \\ \vdots & \vdots & & & \ddots & X \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{matrix}}_{k} \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \ddots & & \vdots \\ \cdots & X & 0 \\ X & \cdots & X \end{matrix}}_{N-1} \underbrace{\begin{matrix} \alpha & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \alpha \\ X & \cdots & X \\ \vdots & & \vdots \\ X & \cdots & X \end{matrix}}_{B-N+1} \right]$$

- $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, all other elements in $\mathbb{F}_q$

Suggested Construction

$\mathbf{H}_1 \Longrightarrow MDS_1$ $\qquad$ $\mathbf{H}_3$

$$\mathbf{G} = \begin{bmatrix} 1 & X & \cdots & X & 0 & 0 & 0 & \cdots & 0 & \alpha & \cdots & 0 \\ 0 & 1 & X & \cdots & X & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 & \cdots & 0 & 0 & \cdots & \alpha \\ \vdots & \vdots & & 1 & \ddots & \ddots & \ddots & & \vdots & X & \cdots & X \\ \vdots & \vdots & & & \ddots & X & \cdots & X & 0 & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 & X & \cdots & X & X & \cdots & X \end{bmatrix}.$$

$\mathbf{H}_2 \Longrightarrow MDS_2$

- $\mathbf{H}_1$ = generator matrix of $(k+N-1, k)$ MDS code over $\mathbb{F}_q$
- $\mathbf{H}_2$ = generator matrix of $(n-(B-N+1), k-(B-N+1))$ MDS code over $\mathbb{F}_q$

Example: $T = 6,\ B = 4,\ N = 3$

Generator matrix

$(6,4)\ MDS_1$

$$G = \begin{bmatrix} 1 & 10 & 9 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 1 & 9 & 1 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 6 & 9 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 4 & 1 & 9 & 8 \end{bmatrix}$$

$(6,2)\ MDS_2$

- $q = 11,\ \alpha \in \mathrm{GF}(121) \setminus \mathrm{GF}(11)$

- Decode s_0
- Trivial when x_0 is not erased $\implies$ assume x_0 is erased

Example: $T = 6,\ B = 4,\ N = 3$

A burst of size $B = 4$ starting at time 0

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

$$
\begin{bmatrix}
1 & 10 & 9 & 0 & 0 & 0 & \alpha & 0 \\
0 & 1 & 9 & 1 & 0 & 0 & 0 & \alpha \\
0 & 0 & 1 & 6 & 9 & 0 & 4 & 8 \\
0 & 0 & 0 & 1 & 4 & 1 & 9 & 8
\end{bmatrix}
$$

- At time 5: s_2 and s_3 are recovered using MDS_2
- At time 6: s_0 is recovered

Example: $T = 6,\ B = 4,\ N = 3$

$N = 3$ sporadic erasures where x_6 is erased

0	1	2	3	4	5	6	7
1	10	9	0	0	0	α	0
0	1	9	1	0	0	0	α
0	0	1	6	9	0	4	8
0	0	0	1	4	1	9	8

- Using MDS_1, all data symbols can be decoded at time 4

Example: $T = 6,\ B = 4,\ N = 3$

$N = 3$ sporadic erasures where x_6 is not erased

0	1	2	3	4	5	6	7
1	10	9	0	0	0	α	0
0	1	9	1	0	0	0	α
0	0	1	6	9	0	4	8
0	0	0	1	4	1	9	8

- $MDS_1^1 =$ "shortening" $\mathbf{H}_1$ by one symbol
  $\Longrightarrow (5,3)$ MDS code with known interference from s_0
- Dashed part of x_6 is in the span of $MDS_1^1 \Longrightarrow$ can be cancelled
- $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \Longrightarrow \alpha$ **is not nulled**
- $\Longrightarrow S_0$ can be recovered

Main Result

Theorem

Block code $\mathcal{C}$ with generator matrix $\mathbf{G}$ is a block code which conforms to $\mathcal{C}(T, B, N)$ with a delay-constraint T and thus a capacity-achieving streaming code of any $\mathcal{C}(T, B, N)$ with delay T and field size that scales quadratically with the delay constraint $(O(T^2))$ can be generated from $\mathcal{C}$ using diagonal interleaving.

- The proof is a generalization of the example

Concluding Remarks

- We show (for the first time):
  An **explicit** capacity achieving construction with field size that scales **quadratically** with the delay constraint
- It can be shown that the generator matrix can be systematic
- Can be used in other applications (broadcast, unequal protection)
- Is it the minimal field size of capacity achieving code?

Thank you for your attention